Graduate Course on Symbolic computation
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Class Notes
Homoological Algebra for the Klein Bottle

1 Introduction

The material presented today is an offshoot of work that was done when I was working at IBM Research, Yorktown Heights, NY with the Scratchpad Group in the 1980s. The system we will be using for this is again, AXIOM. Background material is in [1, 2, 3, 4, 8, 12, 13] which we have already discussed.

2 Homological Perturbation Theory

2.1 The Perturbation Lemma

Recall that given an SDR and, in addition, a second differential d'_Y on Y, let $t = d'_Y - d_Y$. The *perturbation lemma*, states that if we set $t_n = (t\phi)^{n-1}t,\ n \geq 1$ and, for each n, define new maps on X,

$$\partial_n = d + f(t_1 + t_2 + \ldots + t_{n-1})\nabla \quad (1)$$

$$\nabla_n = \nabla + \phi(t_1 + t_2 + \ldots + t_{n-1})\nabla, \quad (2)$$

and on Y:

$$f_n = f + f(t_1 + t_2 + \ldots + t_{n-1})\phi \quad (3)$$

$$\phi_n = \phi + \phi(t_1 + t_2 + \ldots + t_{n-1})\phi, \quad (4)$$

then in the limits, provided they exist, we have new SDR data

$$(X, \partial_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} ((Y, d'_Y), \phi_\infty).$$

More examples relevant to this lecture may be found in the preprint of [11], [10], [8], [9], and other references you already have.

3 The Klein Bottle

The Klein bottle may be described using identifications of the sides of a rectangle as shown in Figure (1). The vertical identifications give a cylinder and if the

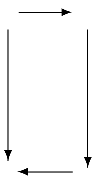

Figure 1: Klein Bottle Construction

horizontal identifications were in the same direction, one would have a torus, but since they are reversed, one has a sort of "twisted torus". Such an object cannot be embedded in three-space, but requires at least four dimensions.

As mentioned above, the Klein Bottle K is a kind of twisted torus. As such, it is a non-trivial circle fibration over a circle:

$$\begin{array}{ccc} S^1 & \longrightarrow & K \\ & & \downarrow \\ & & S^1 \end{array} \tag{5}$$

Taking fundamental groups, one obtains a split exact sequence

$$\mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z} \tag{6}$$

for the fundamental group G of K. Since K is a $K(G,1)$ [15], [16], the homology of G is the homology of K, but we will go further than that in the next section.

4 A resolution of $\mathbb{Z}$ over the integer group ring of G

In this section, a transference problem will be set up. From above, G is a non-trivial semi-direct product of $\mathbb{Z}$ with itself and the only non-trivial morphism from $\mathbb{Z} \longrightarrow Aut(\mathbb{Z})$ is given by $1 \longmapsto (n \longmapsto -n)$, as a group, G is isomorphic to $\mathbb{Z}^2$ with the operation

$$x \cdot y = (x_1 + (-1)^{x_2} y_1, x_2 + y_2) = x + y + P(x,y) \tag{7}$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Z}^2$ and P is a perturbation of the usual free abelian group law.

Let $A = \mathbb{Z}(G)$ and $A_0 = \mathbb{Z}(\mathbb{Z}^2)$. A well-known resolution of $\mathbb{Z}$ over A_0 is given by the Koszul–Tate resolution, where we consider $\mathbb{Z}^2$ as the free abelian group generated by elements t_1 and t_2:

$$A_0 \otimes E[x, y] \xrightarrow{\epsilon} \mathbb{Z}. \tag{8}$$

(one extends the A_0-linear map $x \longmapsto t_1 - 1$ and $y \longmapsto t_2 - 1$ as a graded derivation [17]). It can be shown that one has an SDR

$$A_0 \otimes E[x, y] \underset{f}{\overset{\nabla}{\rightleftarrows}} (A_0 \otimes \bar{B}(A_0), \phi). \tag{9}$$

where $A_0 \otimes \bar{B}(A_0)$ is the bar construction (see [5] for the bar construction). This is a special case of what is constructed in [10].

Now additively, i.e. as $\mathbb{Z}$ modules, $B(A_0) = A_0 \otimes \bar{B}(A_0)$ may be identified with $B(G) = A \otimes \bar{B}(A)$. Thus, we have a transference problem for the SDR (9) using the initiator $t = \partial_G - \partial_{\mathbb{Z}^2}$ where ∂ denotes the differentials in the corresponding bar constructions (both considered as being on $B(A_0)$).

5 The Algorithm and Computation

5.1 The Algorithm

Let z be an element in the Koszul-Tate resolution above and let b_ans_1, t_ans, ans t_phi, and d_inf be in the bar construction above. Also let t_phi $= t\phi$ and d denote the differential in the Tate-Koszul resolution. The following is pseudo-code for the perturbation lemma:

```
d_infty (z) =
    b_ans_1 = nabla z
    t_ans = t (b_ans_1)
    ans = ans + t_ans
    while (not zero? t_ans) repeat
      ans = ans + t_ans
      t_ans = t_phi (t_ans)
    return d(z) + rho(ans)
```

We'll be discussing the complete program details in the next few lectures.

5.2 The Computation

The results of executing the algorithm above give the desired resolution. We have the A-linear differential

$$d(1) = 0 \tag{10}$$
$$d(x) = t_1 - 1 \tag{11}$$
$$d(y) = t_2 - 1 \tag{12}$$
$$d(xy) = (-t_2 - t_1^{-1})x + (-1 + t_1^{-1})y. \tag{13}$$

The reduced differential has only one non-zero term, viz.

$$d(u_1u_2) = -2u_1 \tag{14}$$

which gives the correct homology for the Klein bottle

$$\begin{aligned} H_0(K) &= \mathbb{Z} && (15) \\ H_1(K) &= \mathbb{Z} \oplus \mathbb{Z}/2 && (16) \\ H_n(K) &= 0, \; n > 1. && (17) \end{aligned}$$

Note that one can also obtain the contracting homotopy as well as cycle representatives of the homology classes in the bar construction using this method. More details in general are in [10].

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